HSC Mathematics (2 Unit) Term 1 2007
QUESTION 1 (15 Marks)

Marks

(a) Find $\int (5x+3)^2\,dx$. **2**

(b) Evaluate $\displaystyle\int_1^3 \frac{x^5 - x}{x^2}\,dx$. **3**

(c) (i) Show that $\dfrac{1}{1+e^{-x}} = \dfrac{e^x}{e^x+1}$. **1**

(ii) Hence, or otherwise, find the exact value of $\displaystyle\int_0^1 \frac{1}{1+e^{-x}}\,dx$. **2**

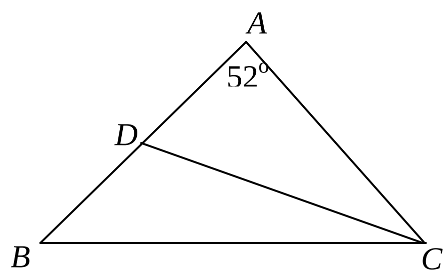
(d) Shade the region, on a number plane, described by the intersection of **2**

$$y \le \sqrt{4-x^2} \quad \text{and} \quad x \ge 0.$$

(e) Find the centre and radius of the circle which has the equation of **2**

$$x^2 + 6x + y^2 - 4y = 12$$

(f) In ΔABC, $AB = AC$, $\angle BAC = 52^\circ$ and CD is drawn so that $\angle ACD = \angle BCD$.



Copy the diagram and find the size of $\angle ADC$, giving reasons. **3**

QUESTION 2 (15 Marks) START A NEW PAGE

(a) Given $f(x) = \sqrt{x^2+4}$.

(i) Copy and complete the table with exact values. **1**

x	0	1	2	3	4
$f(x)$					

(ii) Using the trapezoidal rule, with 5 function values, find an approximation to $\displaystyle\int_0^4 \sqrt{x^2+4}\,dx$, corrected to 2 decimal places. **2**

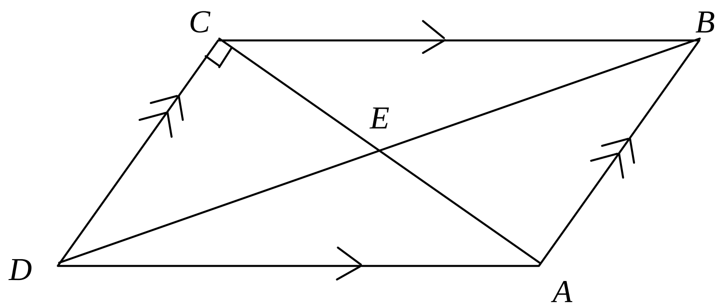
(Question 2 continued) **Marks**

(b) At the beginning of Year 7, Sam decided to study for 5 minutes in the first week of school. In each of the succeeding weeks he increased his study time by 2 minutes per week.

(i) How many hours and minutes will he be studying during week 40? **2**

(ii) Find the total number of hours Sam will have studied by the end of Term 3 in Year 12, where there are 230 school weeks after starting Year 7. **2**

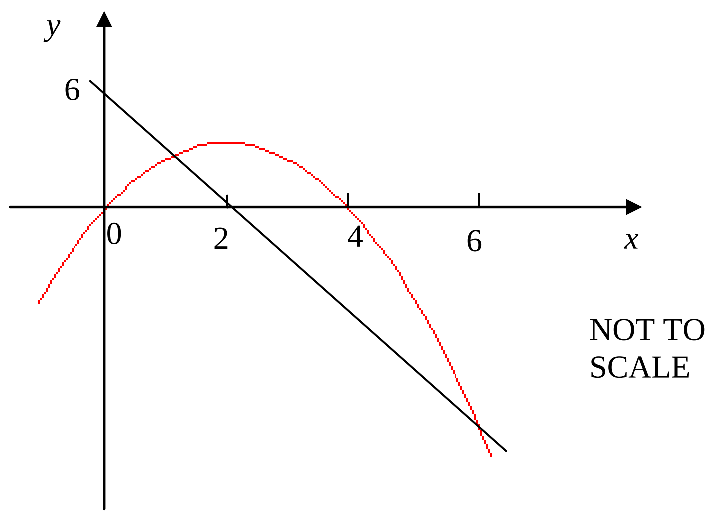
(c) $ABCD$ is a parallelogram. Diagonals AC and BD intersect at E. $\angle ACD = 90^\circ$.



(i) Copy the diagram and show that $AD^2 = DC^2 + 4 \times CE^2$, giving reasons. **2**

(ii) Prove that $AD^2 - DE^2 = 3 \times CE^2$, giving reasons. **1**

(d) The graphs of $y = 4x - x^2$ and $y = 6 - 3x$ are shown below.



(i) Find the x-coordinates of the points of intersection for these graphs. **2**

(ii) Calculate the area bounded by the parabola $y = 4x - x^2$ and the line $y = 6 - 3x$. **3**

QUESTION 3 (15 Marks) START A NEW PAGE

Marks

(a) The area bounded by the curve $y = \dfrac{x^2 - 1}{2}$ and the y-axis, from $y = 1$ to $y = 5$, is rotated about the y-axis. **3**

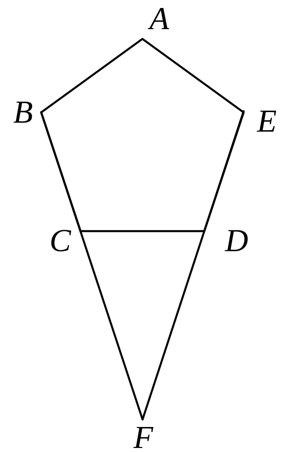
Calculate the volume of the solid formed.

(b) (i) Show that $1 - \dfrac{3}{x+1} = \dfrac{x-2}{x+1}$. **1**

(ii) Sketch, on a number plane, the graph of $y = \dfrac{x-2}{x+1}$. **2**

Label any asymptotes and the x and y intercepts.

(c) *ABCD* is a regular pentagon. Sides *BC* and *ED* are produced to meet at *F*.



(i) Copy the diagram and prove that $CF = DF$, giving reasons. **2**

(ii) Prove that *AF* bisects $\angle BAE$, giving reasons. **3**

(d) John bought a pine-tree plantation which had 100 000 trees on 1st January 2001. At the beginning of each succeeding year (including 2001) he planted 1000 new trees. At the end of each year he removed 5% of the trees growing on the plantation.

Let A_n be the number of trees on the plantation at the end of the n th year.

(i) Show that $A_2 = 100000(0 \cdot 95)^2 + 1000(0 \cdot 95)^2 + 1000(0 \cdot 95)$. **1**

(ii) Hence, show that $A_n = 81000(0 \cdot 95)^n + 19000$. **2**

(iii) Calculate the number of trees that John expects to have at the end of 2020, after he has removed the trees for that year. **1**

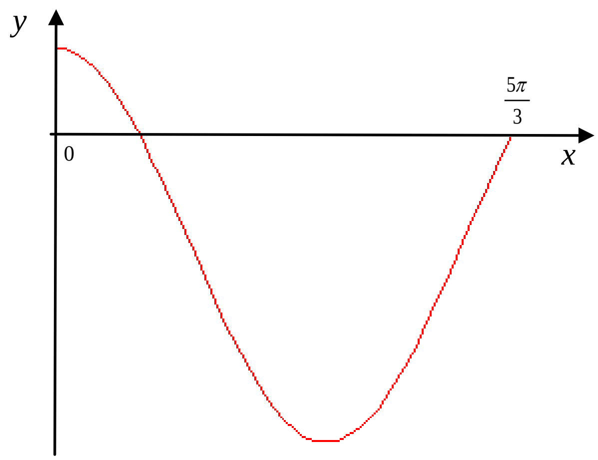
QUESTION 4 (15 Marks) START A NEW PAGE

Marks

(a) The curve $y = f(x)$ has a local minimum at (1,5) and $\dfrac{d^2y}{dx^2} = 6x + 1$. **4**

Find the equation of the curve.

(b) The graph of $y = 2\cos x - 1$ for $0 \le x \le \dfrac{5\pi}{3}$ is shown below.



(i) Show that the graph crosses the x-axis at $x = \dfrac{\pi}{3}$. **1**

(ii) Show that the area enclosed by $y = 2\cos x - 1$, the x-axis and the lines $x = 0$ and $x = \dfrac{5\pi}{3}$ is $(3\sqrt{3} + \pi)$ square units. **4**

(c) The sum of the first n terms of a series is given by $S_n = \dfrac{1}{n(n+1)}$.

(i) Show that $S_{n-1} = \dfrac{1}{(n-1)n}$ for $n \ge 2$. **1**

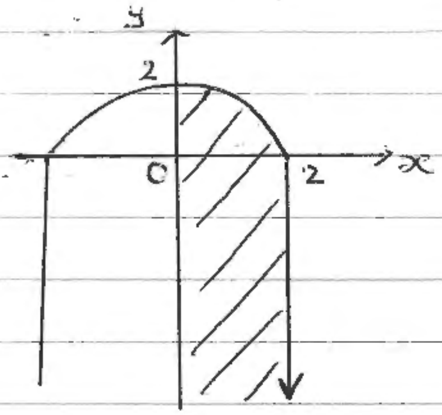
(ii) Show that the n th term (T_n) can be written as $T_n = \dfrac{2}{n(1-n^2)}$ for $n \ge 2$. **2**

(iii) Hence, or otherwise, evaluate $\displaystyle\sum_{n=10}^{30} \frac{1}{n(1-n^2)}$. **3**

END OF EXAMINATION

1(a)	$\int (5x+3)^2 \, dx$	
	$= \dfrac{(5x+3)^3}{3 \times 5} + c$	(1) power (1) denominator
	$= \dfrac{(5x+3)^3}{15} + c$	
1(b)	$\int_1^3 \dfrac{x^5 - x}{x^2} \, dx$	
	$= \int_1^3 \left(x^3 - \dfrac{1}{x}\right) dx$	(1) simplifying
	$= \left[\dfrac{x^4}{4} - \ln x\right]_1^3$	(1) integration
	$= \left(\dfrac{3^4}{4} - \ln 3\right) - \left(\dfrac{1}{4} - \ln 1\right)$	
	$= \dfrac{81}{4} - \dfrac{1}{4} - \ln 3$	
	$= 20 - \ln 3$	(1) answer.
1(c)	(i) $\dfrac{1}{1+e^{-x}} \times \dfrac{e^x}{e^x} = \dfrac{e^x}{e^x+1}$	
	$\int_0^1 \dfrac{1}{1+e^{-x}} \, dx = \int_0^1 \dfrac{e^x}{e^x+1} \, dx$	
	$= \left[\ln(e^x+1)\right]_0^1$	(1) integration
	$= \ln(e+1) - \ln 2$	(1) answer

1(d)



① semicircle

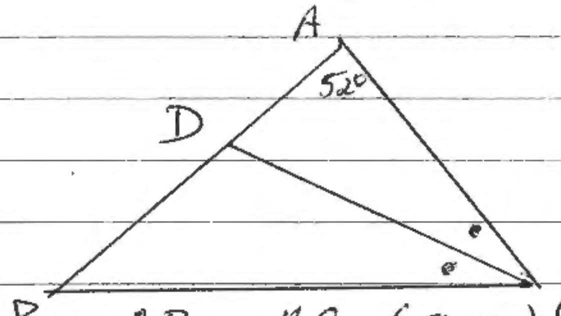
① correct shading

1(e)

$$x^2+6x \quad +y^2-4y=12$$
$$x^2+6x+9 \quad +y^2-4y+4=25$$
$$(x+3)^2+(y-2)^2=5^2 \qquad ① \text{ centre}$$

centre $(-3, 2)$ radius 5 units ① radius

1(f)



$AB = AC$ (given)

$\angle ACB = \angle ABC$ (equal angles opposite equal sides in $\triangle ABC$)

$\angle BAC + \angle ACB + \angle CAB = 180°$ ½

(angle sum of $\triangle ABC = 180°$) ½

$\therefore 2 \times \angle ACB + 52° = 180°$

$\angle ACB = 64°$ ½

$\angle ACD + \angle BCD = 64°$

$\angle ACD = \angle BCD$ (given)

$\therefore \angle ACD = 32°$ ½

$\angle ACD + \angle ADC + \angle BAC = 180°$ ½

(angle sum of $\triangle ACD = 180°$)

$\therefore 32 + 52 + \angle ADC = 180°$

$\therefore \angle ADC = 96°$ ½

(3)

2(a) $f(x) = \sqrt{x^2+4}$

x	0	1	2	3	4
$f(x)$	2	$\sqrt{5}$	$\sqrt{8}$	$\sqrt{13}$	$\sqrt{20}$

① for completely correct table

$\int_0^4 f(x)\,dx =$

$= \frac{1}{2}\left[2 + 2\left[\sqrt{5}+\sqrt{8}+\sqrt{13}\right]+\sqrt{20}\right]$ ① use of trapezoidal rul

$= 11.9061436$

$= 11.91$ (2dP) ① answer (to any numb of dP)

(b) (i) A.P. $a = 5$ $d = 2$ $n = 40$

$t_n = a + (n-1)d$

$= 5 + 39 \times 2$ ①

$= 83$

time = 1 hour 23 mins ①

(ii)

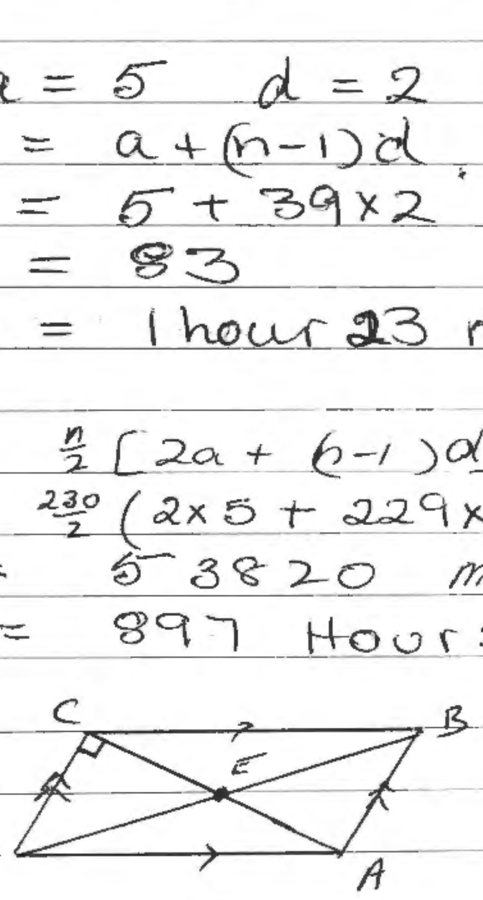
$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{230} = \frac{230}{2}(2\times 5 + 229 \times 2)$ ① correct use o Sn formula

$= 53820$ mins

$= 897$ Hours ① correct no. of hours

(c)



(i) $AD^2 = DC^2 + AC^2$ (Pythag theorem) ①

$2\times CE = AC$ (diagonals of parallelogm bisect eachother) ①

$\therefore AD^2 = DC^2 + (2CE)^2$

$\therefore AD^2 = DC^2 + 4CE^2$

(ii) $DC^2 + CE^2 = DE^2$ (Pythag Theorem) ①

$\therefore AD^2 - DE^2 = 3CE^2$

(by subtraction)

(4)

(d) (i) $y = 4x - x^2$

$y = 6 - 3x$.

$4x - x^2 = 6 - 3x$ ① equation

$\therefore 0 = x^2 - 7x + 6$

$0 = (x-1)(x-6)$

$\therefore x = 1$ or $x = 6$. ① x values

(ii) $A = \int_1^6 [(4x - x^2) - (6-3x)]\,dx$ ① integral

$= \int_1^6 (7x - x^2 - 6)\,dx$.

$= \left[\frac{7x^2}{2} - \frac{x^3}{3} - 6x\right]_1^6$ ① integration

$= \left[\frac{7\times36}{2} - \frac{6^3}{3} - 36\right] - \left[\frac{7}{2} - \frac{1}{3} - 6\right]$

$= 18 - (-2\tfrac{5}{6})$

Area $= 20\tfrac{5}{6}\ u^2$. ① answer

3

(a) $V = \pi \int x^2\,dy$ $\quad y = \frac{x^2 - 1}{2}$.

$2y + 1 = x^2$.

$V = \pi \int_1^5 (2y+1)\,dy$ ① integral

$= \pi \left[\frac{\cancel{2}y^2}{\cancel{2}} + y\right]_1^5$ ① integration

$= \pi\{[25+5] - [1+1]\}$

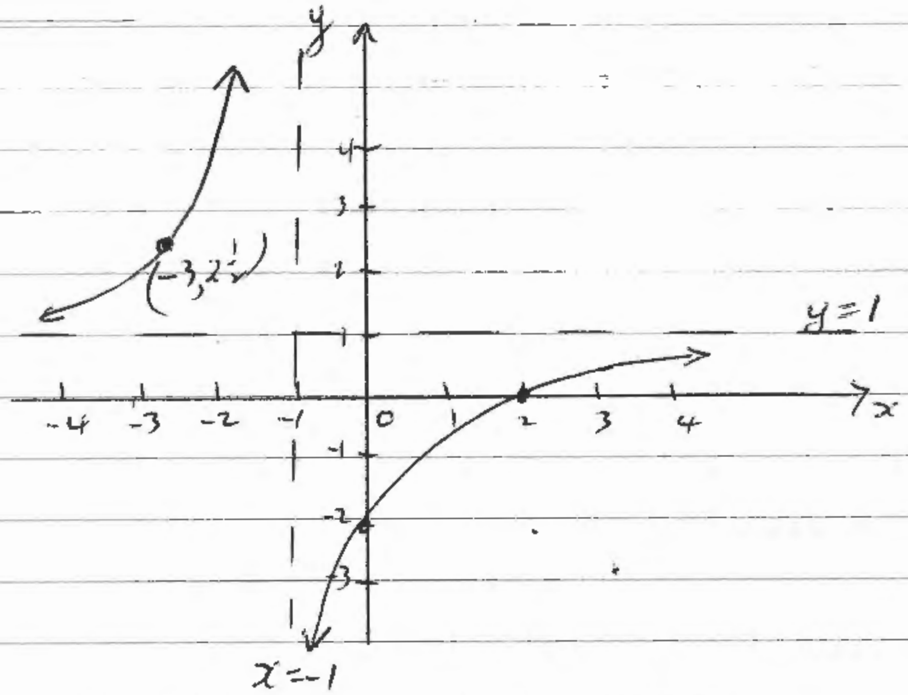
$= \pi \times 28$

Vol $= 28\pi\ u^3$ ① answer

(5)

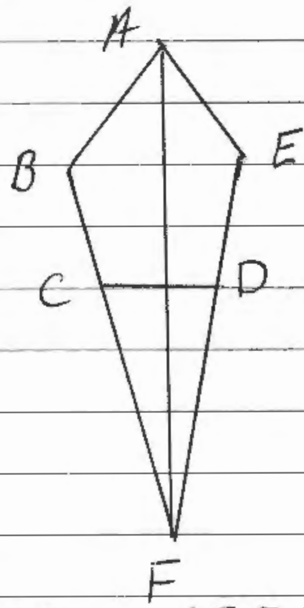
3(b) (ii) $1 - \frac{3}{x+1} = \frac{x+1-3}{x+1}$ (1)

$= \frac{x-2}{x+1}$



$(\frac{1}{2})$ x int / y int
$(\frac{1}{2})$ Vert asy'
$(\frac{1}{2})$ Horiz asy'
$(\frac{1}{2})$ Point $(-3, 2\frac{1}{2})$ or similar / shape, label axes, scale.

(c)



$\angle BCD = \angle EDC$ (equal angles of regular pentagon) $(\frac{1}{2})$

$\angle DCF + \angle BCD = 180°$ (angle sum of straight angle $\angle BCF$) $(\frac{1}{2})$

$\angle CDF + \angle EDC = 180°$ (angle sum of straight angle $\angle EDF$)

$\therefore \angle DCF = \angle CDF$ $(\frac{1}{2})$

$\therefore CF = DF$ (equal angles opposite equal sides in $\triangle CDF$) $(\frac{1}{2})$

(6)

(c)	ii) $BC = ED$ (equal sides of regular pentagon)	$\frac{1}{2}$
	$BF = BC + CF$ $EF = ED + DF$ $\therefore BF = ED$	$\frac{1}{2}$
	$AB = AE$ (equal sides of regular pentagon.	$\frac{1}{2}$
	* AF is common	$\frac{1}{2}$ *
	$\therefore \triangle ABF \equiv \triangle AEF$ (SSS)	$\frac{1}{2}$
	$\therefore \angle BAF = \angle EAF$ (corresponding sides of congruent triangles are equal) $\therefore AF$ bisects $\angle BAE$	$\frac{1}{2}$
	*Alternates.	
	1/ $\angle ABC = \angle AED$ (equal angles in regular pentagon)	$\frac{1}{2}$
	$\therefore \triangle ABF \equiv \triangle AEF$ (SAS)	$\frac{1}{2}$
	2/ ABFE is a kite	$\frac{1}{2}$
	(2 pairs of adjacent equal sides)	$\frac{1}{2}$
	$\therefore AF$ bisects $\angle BAE$ (diagonal which joins vertices of adjacent equal sides, bisects the vertices)	$\frac{1}{2}$.

(d) $A_1 = (100000 + 1000)\,0{\cdot}95$

(i) $A_2 = [(100000 + 1000)(0{\cdot}95) + 1000](0{\cdot}95)$ ①

$= 100000(0{\cdot}95)^2 + 1000(0{\cdot}95)^2 + 1000(0{\cdot}95)$

(ii) $A_n = 100000(0{\cdot}95)^n + 1000(0{\cdot}95)^n + 1000(0{\cdot}95)^{n-1} - - - 1000(0{\cdot}95)$

$= 100000(0{\cdot}95)^n + \dfrac{1000(0{\cdot}95)[1-(.95)^n]}{1-(.95)}$ ① Use of GP Sum formula

$= 100000(0{\cdot}95)^n + 19000(1-(.95)^n)$ ① Calculatio

$= 81000(0{\cdot}95)^n + 19000$

(iii) $n = 20$.

$A_{20} = 81000(0{\cdot}95)^{20} + 19000$

$= 48037{\cdot}35972$ ① answer

$= 48037$ (nearest tree)

4

(a) $\dfrac{d^2y}{dx^2} = 6x + 1$

$\dfrac{dy}{dx} = \dfrac{6x^2}{2} + x + c$ ① primitive

$= 3x^2 + x + c$

$\dfrac{dy}{dx} = 0 \quad x = 1$

$\therefore 0 = 3 + 1 + c$

$\therefore c = -4$ ① c value.

$\dfrac{dy}{dx} = 3x^2 + x - 4$

$y = \dfrac{\cancel{3}x^3}{\cancel{3}} + \dfrac{x^2}{2} - 4x + k$ ① primitive

$x = 1\ y = 5$

$5 = 1 + \frac{1}{2} - 4 + k \quad \therefore k = 7\frac{1}{2}$

$y = x^3 + \frac{1}{2}x^2 - 4x + 7\frac{1}{2}$. ① answer

4. (b)(i) $y = 2\cos x - 1$

$x = \pi/3$

$y = 2\cos\left(\frac{\pi}{3}\right) - 1$

$= 2 \times \frac{1}{2} - 1$

$= 0$ ①

$\therefore\ x = \pi/3\ \ y = 0$ (crosses x axis)

(ii) $A = \int_0^{\pi/3} (2\cos x - 1)\,dx + \left|\int_{\pi/3}^{5\pi/3} (2\cos x - 1)\,dx\right|$ ① correct set of integrals and correct use of absolute.

$= \left[2\sin x - x\right]_0^{\pi/3} + \left|(2\sin x - x)\right|_{\pi/3}^{5\pi/3}$ ① integration and substitution

$= \left(2 \times \sin\frac{\pi}{3} - \frac{\pi}{3}\right) - (0 - 0) + \left|\left(2\sin\frac{5\pi}{3} - \frac{5\pi}{3}\right) - \left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right)\right|$

$= \left(2 \times \frac{\sqrt{3}}{2} - \frac{\pi}{3}\right) + \left|\left(-2 \times \frac{\sqrt{3}}{2} - \frac{5\pi}{3}\right) - \left(\frac{2\times\sqrt{3}}{2} - \frac{\pi}{3}\right)\right|$ ① evaluation of trig values

$= \left(\sqrt{3} - \frac{\pi}{3}\right) + \left|\left(-\sqrt{3} - \frac{5\pi}{3} - \sqrt{3} + \frac{\pi}{3}\right)\right|$

$= \left(\sqrt{3} - \frac{\pi}{3}\right) + \left|\left(-2\sqrt{3} - \frac{4\pi}{3}\right)\right|$ ① working

$= \sqrt{3} - \frac{\pi}{3} + 2\sqrt{3} + \frac{4\pi}{3}$

$= 3\sqrt{3} + \pi$

4 (c)(i)

$$S_n = \frac{1}{n(n+1)}$$

$n \geq 2$

$$S_{n-1} = \frac{1}{(n-1)(n-1+1)}$$

$$= \frac{1}{(n-1)n}$$

① must show some working (substitution)

(ii) $t_n = S_n - S_{n-1}$

$$= \frac{1}{n(n+1)} - \frac{1}{(n-1)n}$$

① formula and correct substitution

$$= \frac{(n-1) - (n+1)}{n(n+1)(n-1)}$$

$$= \frac{-2}{n(n^2-1)}$$

① simplification of algebra.

$$= \frac{2}{n(1-n^2)}$$

(iii)

$$\sum_{n=10}^{30} \frac{1}{n(1-n^2)} = \frac{1}{2} \sum_{n=10}^{30} \frac{2}{n(1-n^2)}$$

① (Difference of Sums)

$$= \frac{1}{2}\left[S_{30} - S_9\right]$$

$$= \frac{1}{2}\left[\frac{1}{30(31)} - \frac{1}{9(10)}\right]$$

① use of correct sums $\frac{1}{2}(S_{30} - S_9)$

$$\frac{1}{2}\left[\frac{1}{930} - \frac{1}{90}\right]$$

$$= \frac{-7}{1395}$$

① Answer.